

ABSTRACT

We prove a formal power series identity, relating the arithmetic sum-of-divisors function to commuting triples of permutations. This establishes a conjecture of Franklin T. Adams-Watters.

A formal identity involving commuting triples of permutations

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The object of this note is to establish the following formal identity:

$$\prod_{j=1}^{\infty} (1 - u^j)^{-\sigma(j)} = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n, \quad (1)$$

where σ is the arithmetic sum-of-divisors function, and $T(n)$ is the number of triples of pairwise-commuting elements of the symmetric group S_n . (Here S_0 is the trivial group.) This is a surprising fact, as there seems no obvious reason for any connection between the function σ and commuting permutations.

The power series expansion of the left-hand side of this identity has coefficients which are listed on the Online Encyclopedia of Integer Sequences (OEIS) [3] as sequence A061256. The coefficients on the right-hand side are listed as sequence A079860. The identity of the two sequences has been stated conjecturally on OEIS. This conjecture (from 2006) is due to Franklin T. Adams-Watters; he informs me that it was based empirically on the numerical evidence.

For a finite group G , we shall write $k(G)$ for the number of conjugacy classes of G . The following simple fact seems first to have been stated by Erdős and Turán [1].

LEMMA 1. *The number of pairs of commuting elements of G is $|G|k(G)$.*

Let $g \in G$. It follows from Lemma 1 that the number of commuting triples of G whose first element is g , is given by $|\text{Cent}_G(g)|k(\text{Cent}_G(g))$. So if $T(G)$ is the total number of commuting triples, then

$$\frac{T(G)}{|G|} = \sum_{g \in G} \frac{|\text{Cent}_G(g)|}{|G|} k(\text{Cent}_G(g)) = \sum_{i=1}^r k(\text{Cent}_G(g_i)), \quad (2)$$

where $\{g_1, \dots, g_r\}$ is a set of conjugacy class representatives for G .

In the case that G is the symmetric group S_n , the conjugacy classes are parameterized by partitions of n , whose parts correspond to cycle lengths. Let $g \in S_n$ have m_t cycles of length t

for all t . Then the centralizer of g in S_n is given (up to isomorphism) by

$$\text{Cent}_{S_n}(g) \cong \prod_{t=1}^n W(t, m_t),$$

where $W(t, m)$ is the wreath product $\mathbf{Z}_t \wr S_m$. (Here \mathbf{Z}_t is used as a shorthand for $\mathbf{Z}/t\mathbf{Z}$, the integers modulo t .) It follows that

$$k(\text{Cent}_{S_n}(g)) = \prod_{t=1}^n k(W(t, m_t)). \quad (3)$$

We may regard an element of $W(t, m)$ as a pair (A, e) , where $A \in \mathbf{Z}_t^m$ and $e \in S_m$. There is a natural action of S_m on the coordinates of \mathbf{Z}_t^m given by $(B^e)_i = B_{ie^{-1}}$. The group multiplication $*$ in $W(t, m)$ is defined by

$$(A, e) * (B, f) = (A + B^e, ef).$$

Conjugacy in groups of the form $H \wr S_m$ is described in [2, Section 4.2]; the case that $H = \mathbf{Z}_t$ is relatively straightforward. Let (A, e) be an element of $W(t, m)$, where $A = (a_1, \dots, a_m)$. Let c be a cycle of the permutation e , and let $\text{supp}(c)$ be the support of c (i.e. the elements of $\{1, \dots, m\}$ moved by c). We shall write $|c|$ for $|\text{supp}(c)|$, the length of the cycle. Define the *cycle sum* $A[c] \in \mathbf{Z}_t$ by

$$A[c] = \sum_{i \in \text{supp}(c)} a_i.$$

The *cycle sum invariant* of (A, e) corresponding to the cycle c is defined to be the pair $(A[c], |c|)$. The element (A, e) has one such invariant for each cycle of e .

LEMMA 2. *Two elements (A, e) and (B, f) of $W(t, m)$ are conjugate in $W(t, m)$ if and only if they have the same cycle sum invariants—that is, if and only if there is a bijection τ between the cycles of e and the cycles of f , such that for any cycle c of e we have $(A[c], |c|) = (B[c\tau], |c\tau|)$.*

Proof. See [2, Theorem 4.2.8], of which this is a particular case. □

Let (A, e) be an element of $W(t, m)$. For each $z \in \mathbf{Z}_t$ we define λ_z to be the partition such that the multiplicity of ℓ as a part of λ_z is equal to the multiplicity of (z, ℓ) as a cycle sum invariant of (A, e) . Lemma 2 tells us that the partitions λ_z for $z \in \mathbf{Z}_t$ determine the conjugacy class of (A, e) in $W(t, m)$. Conversely, a collection of t arbitrary partitions $\{\lambda_z \mid z \in \mathbf{Z}_t\}$ determines a conjugacy class of $W(t, m)$ if and only if the total sum of the sizes of the partitions λ_z is equal to m .

Let $p(d)$ denote the number of partitions of d , and let $P(u)$ be the power series

$$P(u) = \sum_{d=0}^{\infty} p(d)u^d.$$

Consider the formal series

$$Q(u) = \prod_{t=1}^{\infty} P(u^t)^t.$$

From the discussion above, it is easily seen that each monomial term of degree tm in the expansion of $P(u^t)^t$ corresponds to a conjugacy class of $W(t, m)$, and that we therefore have

$$P(u^t)^t = \sum_{m=0}^{\infty} k(W(t, m)) u^{tm}.$$

Now any single term in the expansion of $Q(u)$ corresponds to a choice, firstly of parameters m_t such that $\sum_t tm_t$ is finite, and secondly of a conjugacy class of $W(t, m_t)$ for each t . It follows from (3) that each term of degree n in this expansion corresponds to a conjugacy class of $\text{Cents}_{S_n}(g)$, where g is an element of S_n with m_t cycles of length t . Now by (2) we have the formal identity

$$Q(u) = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n.$$

Thus $Q(u)$ is equal to the right-hand side of (1), and it remains only to show that $Q(u)$ is also equal to the left-hand side.

We use the Eulerian expansion of $P(u)$,

$$P(u) = \prod_{s=1}^{\infty} (1 - u^s)^{-1}.$$

From this it follows that

$$Q(u) = \prod_{t=1}^{\infty} \prod_{s=1}^{\infty} (1 - u^{st})^{-t} = \prod_{j=1}^{\infty} \prod_{t|j} (1 - u^j)^{-t} = \prod_{j=1}^{\infty} (1 - u^j)^{-\sigma(j)},$$

as required.

Finally, I am indebted to Mark Wildon for the observation that both sides of (1) are convergent in the open unit disc $|u| < 1$, and that they therefore represent a complex function which is analytic in this disc. This can be seen by expressing the formal logarithm of the left-hand side of (1) as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sigma(j)}{k} u^{jk} = \sum_{d=1}^{\infty} \left(\sum_{a|d} \frac{a\sigma(a)}{d} \right) u^d,$$

which has radius of convergence 1, since clearly

$$\sum_{a|d} a\sigma(a) < d^4.$$

Thus the left-hand side of (1) represents an analytic function on the disc $|u| < 1$, and it follows that the right-hand side is the Taylor series of that function. An immediate consequence of this observation is that the growth of $T(n)/n!$ is subexponential; I do not know of an easy combinatorial proof of this fact.

References

1. P. Erdős and P. Turán, *On some problems of a statistical group theory IV*, Acta Mathematica Academiae Scientiarum Hungaricae 19 3–4 (1968), 413–435.
2. Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications 16, Addison-Wesley, 1981.
3. The Online Encyclopedia of Integer Sequences, <http://oeis.org/>.

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